

# Biholomorphic Convex Mappings of Order $\alpha$ on the Unit Ball in Hilbert Spaces

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**Abstract.** In this paper, we first introduce the concept of biholomorphic convex mapping of order  $\alpha$  on the unit ball  $B$  in a complex Hilbert space  $X$ . Next we provide some sufficient conditions that a locally biholomorphic mapping  $f$  is a biholomorphic convex mapping of order  $\alpha$  and give an Alexander's theorem between the subclass of convex mappings and the subclass of starlike mappings on  $B$  in Hilbert space. We also obtain the order of starlikeness of biholomorphic convex mappings of order  $\alpha$  on  $B$  in Hilbert spaces. Finally, we construct some concrete examples of biholomorphic convex mappings of order  $\alpha$  on  $B$  in Hilbert spaces by means of a linear operator.

**Keywords.** Biholomorphic convex mapping; Biholomorphic starlike mapping; Locally biholomorphic mapping; biholomorphic convex mapping of order  $\alpha$ .

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## 1 Introduction

The holomorphic functions of one complex variable which map the unit disk onto starlike or convex domains have been extensively studied. These functions are easily characterized by simple analytic or geometric conditions. In moving to higher dimensions, several difficulties arise. In the case of one complex variable, the following well known theorems had been established(cf. [3]).

**Theorem A** Suppose that  $\alpha \in [0, 1)$  and  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  is a holomorphic function on the unit disk  $U = \{z \in \mathbb{C} : |z| < 1\}$  in the complex plane  $\mathbb{C}$ .

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- (1) If  $\sum_{n=2}^{\infty} n^2 |a_n| \leq 1$ , then  $f$  is a convex function in the unit disk  $U$ .
- (2) If  $\sum_{n=2}^{\infty} n(n-\alpha) |a_n| \leq 1-\alpha$ , then  $f$  is a convex function of order  $\alpha$  in  $U$ .

**Theorem B** Suppose that  $\alpha \in [0, 1)$  and  $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$  is a holomorphic function on the unit disk  $U$  in the complex plane  $\mathbb{C}$ .

- (1) If  $\sum_{n=2}^{\infty} n |a_n| \leq 1$ , then  $f$  is a starlike function in the unit disk  $U$ .
- (2) If  $\sum_{n=2}^{\infty} (n-\alpha) |a_n| \leq 1-\alpha$ , then  $f$  is a starlike function of order  $\alpha$  in  $U$ .

Roper and Suffridge established the  $n$ -dimensional version of Theorem A(1), and we[9] established the  $n$ -dimensional version of Theorem B(1)(2) as follows.

**Theorem C**(Roper and Suffridge[12]) Let  $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$  be a holomorphic mapping on the unit ball  $B_2^n$ . If  $\sum_{k=2}^{\infty} k^2 \|A_k\| \leq 1$ , then  $f(z)$  is a convex mapping on  $B_2^n$ .

**Theorem D**(Liu and Zhu[9]) Suppose that  $\alpha \in [0, 1)$ . Let  $f(z) = z + \sum_{k=2}^{\infty} A_k(z^k)$  be a holomorphic mapping on the unit ball  $B$  in Hilbert space. If  $\sum_{k=2}^{\infty} (k-\alpha) \|A_k\| \leq A(\alpha)$ , where  $A(\alpha)$  is defined by

$$A(\alpha) = \begin{cases} \frac{(2-\alpha)\sqrt{1-2\alpha}}{\sqrt{5-2\alpha}}, & 0 \leq \alpha \leq \frac{1}{4}, \\ \frac{(2-\alpha)(1-\alpha)}{2+\alpha}, & \frac{1}{4} < \alpha \leq \frac{2}{5}, \\ \alpha, & \frac{2}{5} < \alpha < \frac{1}{2}, \\ 1-\alpha, & \frac{1}{2} \leq \alpha < 1. \end{cases}$$

Then  $f(z)$  is a starlike mapping of order  $\alpha$  on  $B$  in Hilbert space.

A problem is naturally proposed: can we establish the  $n$ -dimensional version for Theorem A(2)?

## 2 Preliminaries

In order to state and prove our main results, we recall some definitions and notations. Suppose that  $X$  is a complex Hilbert space with inner product  $\langle \cdot, \cdot \rangle$  and norm  $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$ , and  $G$  is a domain in  $X$ . A mapping  $f : G \rightarrow X$  is said to be holomorphic on  $G$ , if for any  $z \in G$ , there exists a linear operator  $Df(z) : X \rightarrow X$  such that

$$\lim_{h \rightarrow 0} \frac{\|f(z+h) - f(z) - Df(z)h\|}{\|h\|} = 0.$$

The linear operator  $Df(z)$  is called the Fréchet derivative of  $f$  at  $z \in G$ .

If  $f$  is holomorphic on  $G$ , then for every  $k = 1, 2, \dots$ , and every  $z_0 \in G$ , there is a bounded symmetric  $k$ -linear operator  $D^k f(z_0) : X \times X \times \dots \times X \rightarrow X$  such that

$$f(z) = \sum_{k=0}^{+\infty} \frac{1}{k!} D^k f(z_0)((z - z_0)^k)$$

for all  $z$  in some neighborhood of  $z_0$ , where  $D^0 f(z_0)((z - z_0)^0) = f(z_0)$  and

$$D^k f(z_0)((z - z_0)^k) = D^k f(z_0)(z - z_0, z - z_0, \dots, z - z_0)$$

for  $k \geq 1$ .

A mapping  $f : G \rightarrow X$  is said to be biholomorphic on  $G$  if  $f$  is holomorphic on  $G$ ,  $f(G)$  is a domain, and the inverse  $f^{-1}$  exists and is holomorphic on  $f(G)$ . A mapping  $f : G \rightarrow X$  is said to be locally biholomorphic on  $G$ , if for any  $z \in G$ , there exists a neighborhood  $U$  of  $z$  such that  $f|_U$  is biholomorphic on  $U$ . Then  $f$  is locally biholomorphic on  $G$  if and only if its Fréchet derivative  $Df(z)$  has a bounded inverse at each  $z \in G$ .

The unit ball in  $X$  is  $B = \{z \in X : \|z\| < 1\}$ . Let  $N(B)$  denote the class of all local biholomorphic mappings  $f : B \rightarrow X$  such that  $f(0) = 0, Df(0) = I$ , where  $I$  is the identity operator in  $X$ . A biholomorphic mapping  $f : B \rightarrow X$  is called a biholomorphic starlike mapping if  $tf(B) \subset f(B)$  for  $0 \leq t \leq 1$  with  $f(0) = 0$ . Let  $S^*(B)$  be the subclass of  $N(B)$  consisting of starlike mappings on  $B$ . Then  $f \in S^*(B)$  if and only if  $f$  is locally biholomorphic such that

$$\operatorname{Re} \langle Df(z)^{-1} f(z), z \rangle > 0$$

for all  $z \in B \setminus \{0\}$  (cf. [1, 3, 6]).

A mapping  $f \in N(B)$  is called starlike of order  $\alpha \in (0, 1)$  on  $B$  if

$$\left| \langle Df(z)^{-1} f(z), z \rangle - \frac{1}{2\alpha} \|z\|^2 \right| < \frac{1}{2\alpha} \|z\|^2 \quad \text{for all } z \in B \setminus \{0\},$$

Let  $S^*(B, \alpha)$  denote the class of starlike mappings of order  $\alpha$  on  $B$  for  $0 < \alpha < 1$  and let  $S^*(B, 0) \equiv S^*(B)$ . It is obvious that  $S^*(B, \alpha) \subset S^*(B)$  for  $0 \leq \alpha < 1$ .

A biholomorphic mapping  $f : B \rightarrow X$  is said biholomorphic convex mapping if

$$(1 - t)f(z_1) + tf(z_2) \in f(B)$$

for all  $z_1, z_2 \in B$  and  $0 \leq t \leq 1$ . The class of all biholomorphic convex mappings on  $B$  with  $f(0) = 0, Df(0) = I$  is denoted by  $K(B)$ .

We[16] obtained a necessary and sufficient condition that a locally biholomorphic mapping was a biholomorphic convex mapping on  $B$  in the Hilbert space  $X$  as follows.

**Theorem A (Zhu and Liu[16]).** Let  $f : B \rightarrow X$  be a locally biholomorphic mapping. Then  $f$  is a biholomorphic convex mapping on  $B$  if and only if

$$\|x\|^2 - \operatorname{Re} \langle Df(z)^{-1} D^2 f(z)(x, x), z \rangle > 0 \quad (2.1)$$

for  $z \in B \setminus \{0\}$  and  $x \in X \setminus \{0\}$  with  $\operatorname{Re}\langle x, z \rangle = 0$ .

**Remark 1.** Theorem A had improved a result of Hamada and Kohr[4]. Setting  $X = C^n$  in Theorem A, we also obtain Theorem 2 in [2].

**Corollary 1.** Let  $0 \leq \alpha < 1$  and  $f : B \rightarrow X$  be a locally biholomorphic mapping. If  $f$  satisfies the following inequality

$$\|x\|^2 - \operatorname{Re}\langle Df(z)^{-1}D^2f(z)(x, x), z \rangle > \alpha \cdot \|x\|^2 \quad (2.2)$$

for  $z \in B \setminus \{0\}$  and  $x \in X \setminus \{0\}$  with  $\operatorname{Re}\langle x, z \rangle = 0$ . Then  $f$  is a biholomorphic convex mapping on  $B$ .

We call such mapping  $f$ , which satisfies the hypothesis of Corollary 1, a biholomorphic convex mapping of order  $\alpha$  on  $B$ . We let  $K(B, \alpha)$  denote the subclass of all biholomorphic convex mappings of order  $\alpha$  on  $B$  with  $f(0) = 0, Df(0) = I$ .

In this paper, we provide some sufficient conditions for biholomorphic convex mapping of order  $\alpha$  and an Alexander's theorem between the subclass of convex mappings and the subclass of starlike mappings on  $B$  in Hilbert space. We also obtain the order of starlikeness of biholomorphic convex mappings of order  $\alpha$  on  $B$  in Hilbert spaces. Finally, we introduce a linear operator in purpose to construct some concrete examples of biholomorphic convex mappings on  $B$  in Hilbert spaces. From these, we give some examples of biholomorphic convex mappings on  $B$  in Hilbert spaces.

### 3 Main Results

We first establish some sufficient conditions for biholomorphic convex mapping of order  $\alpha$  on  $B$ .

**Theorem 1.** Suppose that  $0 \leq \alpha < 1$  and  $f : B \rightarrow X$  is a locally biholomorphic mapping. If  $f$  satisfies

$$\|Df(z)^{-1}D^2f(z)(x, x)\| \leq 1 - \alpha \quad (3.1)$$

for  $z \in B \setminus \{0\}$  and  $x \in X$  with  $\|x\| = 1$  and  $\operatorname{Re}\langle x, z \rangle = 0$ , then  $f$  is a biholomorphic convex mapping of order  $\alpha$  on  $B$ .

**Proof.** Since  $f : B \rightarrow X$  is a locally biholomorphic mapping, for any  $z \in B \setminus \{0\}$  and  $x \in X \setminus \{0\}$  with  $\|x\| = 1$  and  $\operatorname{Re}\langle x, z \rangle = 0$ , we have

$$\begin{aligned} \|x\|^2 - \operatorname{Re}\langle Df(z)^{-1}D^2f(z)(x, x), z \rangle &\geq \|x\|^2 - |\langle Df(z)^{-1}D^2f(z)(x, x), z \rangle| \\ &\geq \|x\|^2 - \|Df(z)^{-1}D^2f(z)(x, x)\| \|z\| \\ &> \|x\|^2 - \|Df(z)^{-1}D^2f(z)(x, x)\| \\ &= \|x\|^2 - \|Df(z)^{-1}D^2f(z)(\frac{x}{\|x\|}, \frac{x}{\|x\|})\| \|x\|^2. \end{aligned}$$

Notice that  $\|\frac{x}{\|x\|}\| = 1$ , we conclude from (3.1) that

$$\begin{aligned} \|x\|^2 - \operatorname{Re}\langle Df(z)^{-1}D^2f(z)(x, x), z \rangle &> \|x\|^2 - \|Df(z)^{-1}D^2f(z)(\frac{x}{\|x\|}, \frac{x}{\|x\|})\| \|x\|^2 \\ &\geq \alpha \cdot \|x\|^2. \end{aligned}$$

Hence by Corollary 1, we obtain that  $f \in K(B, \alpha)$ , and the proof is complete.

**Corollary 2.** Suppose that  $0 \leq \alpha < 1$  and  $f : B \rightarrow X$  is a locally biholomorphic mapping with  $\|Df(z) - I\| \leq c < 1$  for each  $z \in B$ , where  $I$  is the identity operator in  $X$ . If  $f$  satisfies

$$\|D^2f(z)(x, x)\| \leq (1 - c)(1 - \alpha)$$

for all  $x \in X$  with  $\|x\| = 1$  and  $z \in B \setminus \{0\}$  such that  $\operatorname{Re}\langle x, z \rangle = 0$ , then  $f$  is a biholomorphic convex mapping of order  $\alpha$  on  $B$ .

**Proof.** Since  $\|Df(z) - I\| \leq c < 1$  for any  $z \in B$ , we obtain that  $Df(z) = I - (I - Df(z))$  is an invertible linear operator (see [13], P192), and

$$\|Df(z)^{-1}\| \leq \frac{1}{1 - \|I - Df(z)\|} \leq \frac{1}{1 - c}$$

for all  $z \in B$ .

Thus for any  $x \in X$  with  $\|x\| = 1$  and  $z \in B \setminus \{0\}$  such that  $\operatorname{Re}\langle x, z \rangle = 0$ , by the hypothesis of Corollary 2, we have

$$\begin{aligned} \|Df(z)^{-1}D^2f(z)(x, x)\| &\leq \|Df(z)^{-1}\| \|D^2f(z)(x, x)\| \\ &\leq \frac{1}{1 - c} \cdot (1 - c)(1 - \alpha) = 1 - \alpha. \end{aligned}$$

Hence by Theorem 1, we obtain that  $f$  is a biholomorphic convex mapping of order  $\alpha$  on  $B$ , and the proof is complete.

**Remark 2.** Setting  $\alpha = 0$  in Theorem 1, we get Corollary 1 in [16]; Setting  $\alpha = 0$  in Corollary 2, we get Corollary 2 in [16].

**Theorem 2.** Let  $0 \leq \alpha < 1$  and  $f(z) = z + \sum_{k=2}^{+\infty} A_k(z^k) : B \rightarrow X$  be a holomorphic mapping. If  $f$  satisfies  $\sum_{k=2}^{+\infty} k(k - \alpha)\|A_k\| \leq 1 - \alpha$ , then  $f \in K(B, \alpha)$ .

**Proof.** By direct calculating the Fréchet derivatives of  $f(z)$ , we obtain

$$\begin{aligned} Df(z) &= I + \sum_{k=2}^{+\infty} kA_k(z^{k-1}, \cdot), \\ D^2f(z)(x, x) &= \sum_{k=2}^{+\infty} k(k - 1)A_k(z^{k-2}, x^2) \end{aligned}$$

and

$$\|Df(z) - I\| \leq \sum_{k=2}^{+\infty} k\|A_k\| \leq \frac{1}{2 - \alpha} \sum_{k=2}^{+\infty} k(k - \alpha)\|A_k\| \leq \frac{1 - \alpha}{2 - \alpha} < 1$$

for  $z \in B$ ,  $x \in X$ . Hence we obtain that  $Df(z) = I - (I - Df(z))$  is an invertible

linear operator (see [13], P192), and

$$\begin{aligned}
\|D^2 f(z)(x, x)\| &\leq \sum_{k=2}^{+\infty} (k^2 - k) \|A_k\| \|z\|^{k-2} \|x\|^2 \\
&\leq 1 - \alpha + \alpha \sum_{k=2}^{+\infty} k \|A_k\| - \sum_{k=2}^{+\infty} k \|A_k\| \\
&= (1 - \sum_{k=2}^{+\infty} k \|A_k\|)(1 - \alpha)
\end{aligned}$$

for  $z \in B \setminus \{0\}$  and  $\|x\| = 1$  with  $\operatorname{Re}\langle x, z \rangle = 0$ . By Corollary 2 for  $c = \sum_{k=2}^{+\infty} k \|A_k\|$ , we obtain that  $f \in K(B, \alpha)$ , and the proof is complete.

**Remark 3.** Setting  $X = \mathbb{C}^n$ ,  $\alpha = 0$  in Theorem 2, we may obtain Theorem 2.1 in [12]. Our proof is more simple than theirs. Setting  $X = \mathbb{C}$  in Theorem 2, we get Theorem A(2).

**Example 1.** Let  $0 \leq \alpha < 1$  and  $A$  be a symmetric bilinear operator from  $X \times X$  to  $X$  with  $\|A\| \leq \frac{1-\alpha}{4-2\alpha}$ . If we let  $f(z) = z + A(z, z)$ , then  $f \in K(B, \alpha)$ .

**Proof.** Some straightforward computations yield the relations

$$Df(z) = I + 2A(z, \cdot), \quad D^2 f(z)(x, y) = 2A(x, y)$$

for  $z \in B$ ,  $x, y \in X$ . It implies

$$Df(0) = I, \quad D^2 f(0)(\cdot, \cdot) = 2A(\cdot, \cdot) \quad \text{and} \quad D^k f(0) = 0$$

for  $k = 3, 4, \dots$ . Hence we obtain

$$\sum_{k=2}^{+\infty} \frac{k(k-\alpha) \|D^k f(0)\|}{k!} = (2-\alpha) \|D^2 f(0)\| = 2(2-\alpha) \|A\| \leq 1 - \alpha.$$

By Theorem 2, we conclude that  $f \in K(B, \alpha)$ , and the proof is complete.

**Example 2.** Let  $0 \leq \alpha < 1$ ,  $0 < |a| \leq 1/2$  and  $u \in X$  with  $\|u\| = 1$ . Then

$$f(z) = z + a\langle z, u \rangle^2 u \in K(B, \alpha) \iff |a| \leq \frac{1-\alpha}{4-2\alpha}.$$

**Proof.** Let  $c = 1 - \frac{2|a|}{1-\alpha}$ . If  $0 < |a| \leq \frac{1-\alpha}{4-2\alpha}$ , then we have  $\frac{1-\alpha}{2-\alpha} \leq c < 1$  and  $2|a| = (1-c)(1-\alpha) \leq c$ . Short computations yield the relations

$$Df(z) = I + 2a\langle z, u \rangle \langle \cdot, u \rangle u, \quad D^2 f(z)(x, x) = 2a\langle x, u \rangle^2 u. \quad (3.2)$$

It implies

$$\|Df(z) - I\| \leq 2|a| \|z\| < 2|a| \leq c, \quad \|D^2 f(z)(x, x)\| \leq 2|a| = (1-c)(1-\alpha)$$

for all  $x \in X$  with  $\|x\| = 1$  and  $z \in B$  such that  $\operatorname{Re}\langle x, z \rangle = 0$ .

By Corollary 2, we obtain that  $f \in K(B, \alpha)$ .

Conversely, we shall prove that  $0 < |a| \leq \frac{1-\alpha}{4-2\alpha}$  when  $f \in K(B, \alpha)$ .

If not, then  $|a| > \frac{1-\alpha}{4-2\alpha}$ . Let  $\theta = \arg a$ ,  $x = ie^{-i\theta}u$  and  $z_0 = -re^{-i\theta}u$  for  $\frac{1-\alpha}{(4-2\alpha)|a|} < r < 1$ , where  $u \in X$  with  $\|u\| = 1$ . Then  $\|x\| = 1$ ,  $z_0 \in B \setminus \{0\}$  and  $\operatorname{Re}\langle x, z_0 \rangle = \operatorname{Re}\{-ir\} = 0$ .

Some straightforward computations from (3.2) yield the relations

$$\begin{aligned} Df(z_0)^{-1} &= I + \frac{2|a|r}{1-2|a|r}\langle \cdot, u \rangle u, \quad D^2f(z_0)(x, x) = -2|a|e^{-i\theta}u \\ Df(z_0)^{-1}D^2f(z_0)(x, x) &= -\frac{2|a|e^{-i\theta}}{1-2|a|r}u. \end{aligned}$$

Hence we obtain

$$\|x\|^2 - \operatorname{Re}\langle Df(z_0)^{-1}D^2f(z_0)(x, x), z_0 \rangle = \frac{1-4|a|r}{1-2|a|r} < \alpha.$$

This contradicts (2.2), and the proof is complete.

Next, we provide an Alexander's theorem between the subclass of convex mappings and the subclass of starlike mappings on  $B$  in Hilbert space. In the case of one complex variable, Alexander's theorem told us that  $f(z)$  is a convex function on the unit disc  $U$  if and only if  $zf'(z)$  is a starlike function on the unit disc  $U$ . This theorem is no longer true in several complex variables (see [3]). However, we have the following Alexander's theorem.

**Theorem 3(Alexander's Theorem).** Suppose that  $0 \leq \alpha < 1$  and  $A(\alpha)$  is defined by Theorem D. Let

$$SK(B, \alpha) = \{f(z) : f(z) = z + \sum_{k=2}^{\infty} A_k(z^k) \in H(B) \text{ such that } \sum_{k=2}^{\infty} k(k-\alpha) \|A_k\| \leq A(\alpha)\},$$

and

$$SS^*(B, \alpha) = \{f(z) : f(z) = z + \sum_{k=2}^{\infty} A_k(z^k) \in H(B) \text{ such that } \sum_{k=2}^{\infty} (k-\alpha) \|A_k\| \leq A(\alpha)\}.$$

Then  $SK(B, \alpha) \subset K(B)$ ,  $SS^*(B, \alpha) \subset S^*(B)$ , and  $f(z) \in SK(B, \alpha)$  if and only if  $Df(z)(z) \in SS^*(B, \alpha)$ .

Notice that  $A(\alpha) \leq 1 - \alpha$  for  $0 \leq \alpha < 1$ , by applying Theorem D and Theorem 2, we can prove this theorem easily.

Now we establish a result on the order of starlikeness of function class  $K(B, \alpha)$ .

**Theorem 4.** Suppose that  $0 \leq \alpha < 1$ . Then  $K(B, \alpha) \subset S^*(B, \beta)$ , where

$$\beta = \beta(\alpha) = \frac{2\alpha - 1 + \sqrt{(2\alpha - 1)^2 + 8}}{4}.$$

In order to prove the above theorem, we need the following lemmas.

**Lemma 1.** ([10]) Let  $g(z) = a + a_1z + \cdots$  is analytic in  $U$  and  $g(z) \not\equiv a$ . If there exists  $z_0 \in U \setminus \{0\}$  such that  $|g(z_0)| = \max_{|z| \leq |z_0|} |g(z)|$ , then there exists real number  $t \geq \frac{|g(z_0)| - |a|}{|g(z_0)| + |a|}$  such that  $z_0 g'(z_0) = t g(z_0)$ .

**Lemma 2.** Let  $f \in N(B)$ ,  $0 < \rho < 1$ . If there exists  $z_0 \in B \setminus \{0\}$  such that

$$\operatorname{Re} \frac{\|z\|^2}{\langle Df(z_0)^{-1}f(z_0), z_0 \rangle} = \rho,$$

and  $\operatorname{Re} \frac{\|z\|^2}{\langle Df(z)^{-1}f(z), z \rangle} \geq \rho$  for all  $\|z\| < \|z_0\|$ . Then there exist real numbers  $\theta, t, m$  such that:

- (i)  $\langle h(z_0), z_0 \rangle = \frac{\|z_0\|^2}{2\rho} (1 + e^{i\theta})$ , where  $h(z) = Df(z)^{-1}f(z)$  ;
- (ii)  $\langle Dh(z_0)(z_0), z_0 \rangle = \frac{\|z_0\|^2}{2\rho} [1 + (1+t)e^{i\theta}]$ , where  $t \geq \frac{1-|2\rho-1|}{1+|2\rho-1|}$  ;
- (iii)  $e^{i\theta} \overline{Dh(z_0)}(z_0) + e^{-i\theta} h(z_0) = mz_0$ , where  $m = \frac{2\cos\theta + 2 + t}{2\rho}$ .

**Proof.** Since  $h : B \rightarrow X$  is a holomorphic mapping and  $h(0) = 0, Dh(0) = I$ , and

$$\left| \langle h(z_0), z_0 \rangle - \frac{\|z_0\|^2}{2\rho} \right| = \frac{\|z_0\|^2}{2\rho}, \quad (3.3)$$

$$\left| \langle h(z), z \rangle - \frac{\|z\|^2}{2\rho} \right| \leq \frac{\|z\|^2}{2\rho}, \quad \|z\| < \|z_0\|. \quad (3.4)$$

Let

$$\psi(\xi) = \frac{2\rho}{\|\xi z_0\|^2} \langle h(\xi z_0), \xi z_0 \rangle - 1 = \frac{2\rho}{\|z_0\|^2} \left\langle \frac{h(\xi z_0)}{\xi}, z_0 \right\rangle - 1,$$

then  $\psi(\xi)$  is analytic in  $\overline{U}$  and  $|\psi(\xi)| \leq 1 = |\psi(1)|$  for  $\xi \in \overline{U}$ ,  $\psi(0) = 2\rho - 1$ . By Lemma 1, there is a real number  $t \geq \frac{1-|2\rho-1|}{1+|2\rho-1|}$  such that  $\psi'(1) = t\psi(1)$ .

Let  $\psi(1) = e^{i\theta}$  for some real number  $\theta$ , then we obtain (i) holds, and

$$\psi'(1) = \frac{2\rho}{\|z_0\|^2} \langle Dh(z_0)(z_0) - h(z_0), z_0 \rangle = te^{i\theta},$$

which implies (ii) holds.

From (3.3) and (3.4), we obtain that

$$\operatorname{Re}[e^{-i\theta} \langle h(z), z \rangle] \leq \frac{\|z_0\|^2}{2\rho} (1 + \cos \theta) = \operatorname{Re}[e^{-i\theta} \langle h(z_0), z_0 \rangle], \quad \|z\| < \|z_0\|. \quad (3.5)$$

Let  $r = \|z_0\|$ ,  $B(r) = \{z \in X : \|z\| < r\}$ , then the tangent hyperplane of  $\partial B(r)$  at  $z_0$  is

$$T_{z_0}(\partial B(r)) = \{b \in X : \langle b, z_0 \rangle = 0\}.$$

For any tangent vector  $a \in T_{z_0}(\partial B(r))$  with  $\|a\| = 1$ , set  $\gamma(t) = \sqrt{1-t^2} z_0 + t ra$ , then  $\|\gamma(t)\| = r$  for  $t \in (-1, 1)$  and  $\gamma(0) = z_0, \gamma'(0) = ra$ . Let

$$\varphi(t) = \operatorname{Re}[e^{-i\theta} \langle h(\gamma(t)), \gamma(t) \rangle].$$



From (3.5), we obtain  $\varphi(t) \leq \varphi(0)$  for  $t \in (-1, 1)$ , so that  $\varphi(0) = \max_{t \in (-1, 1)} \varphi(t)$ . Hence

$$0 = \varphi'(0) = \operatorname{Re}[e^{-i\theta} \langle Dh(z_0)ra, z_0 \rangle + e^{-i\theta} \langle h(z_0), ra \rangle] = r \operatorname{Re} \langle v, a \rangle,$$

where  $v = e^{i\theta} \overline{Dh(z_0)}(z_0) + e^{-i\theta} h(z_0)$ . This implies that  $v$  is a normal vector of  $\partial B(r)$  at  $z_0$ , thus there exists a real number  $m$  such that  $v = mz_0$ . Since

$$\begin{aligned} m\|z_0\|^2 &= \operatorname{Re} \langle z_0, mz_0 \rangle = \operatorname{Re} \langle z_0, v \rangle \\ &= \operatorname{Re} \langle z_0, e^{i\theta} \overline{Dh(z_0)}(z_0) \rangle + \operatorname{Re} \langle z_0, e^{-i\theta} h(z_0) \rangle \\ &= \operatorname{Re} [e^{-i\theta} \langle Dh(z_0)(z_0), z_0 \rangle] + \operatorname{Re} \langle [e^{-i\theta} h(z_0), z_0] \rangle \\ &= \frac{\|z_0\|^2}{2\rho} (\cos \theta + 1 + t) + \frac{\|z_0\|^2}{2\rho} (\cos \theta + 1), \end{aligned}$$

we obtain that  $m = \frac{2\cos\theta+2+t}{2\rho}$ , and this completes the proof of Lemma 2.

**Proof of Theorem 4.** Let  $h(z) = [Df(z)]^{-1}f(z)$  with  $f(z) \in K(B, \alpha)$ , and  $g(z) = \frac{\|z\|^2}{\langle h(z), z \rangle}$ , then  $g(0) = 1 > \beta = \beta(\alpha)$ .

Suppose  $f \notin S^*(B, \beta)$ , then by the continuity of  $g(z)$ , there exists  $z_0 \in B \setminus \{0\}$  such that  $\operatorname{Re} g(z_0) = \beta$  and  $\operatorname{Re} g(z) \geq \beta$  for all  $\|z\| < \|z_0\|$ . Thus it follows from Lemma 2 that there exist real numbers  $\theta, t \geq \frac{1-\beta}{\beta}$  and  $m = \frac{2\cos\theta+2+t}{2\beta}$  such that (i)-(iii) of Lemma 2 hold.

Let  $b = e^{-i\theta} h(z_0) - \frac{z_0}{2\beta}(1 + \cos \theta)$ , then it follows from Lemma 2(i) that

$$\operatorname{Re} \langle b, z_0 \rangle = \operatorname{Re} [e^{-i\theta} \langle h(z_0), z_0 \rangle] - \frac{\|z_0\|^2}{2\beta}(1 + \cos \theta) = 0,$$

so that

$$\operatorname{Re} \langle [Df(z_0)]^{-1} D^2 f(z_0)(b, b), z_0 \rangle < (1 - \alpha) \|b\|^2. \quad (3.6)$$

Let  $b_1 = iz_0$ , then  $\operatorname{Re} \langle b_1, z_0 \rangle = \operatorname{Re}[i\|z_0\|^2] = 0$ , so we conclude from the fact  $f(z) \in K(B, \alpha)$  that

$$\operatorname{Re} \langle [Df(z_0)]^{-1} D^2 f(z_0)(iz_0, iz_0), z_0 \rangle < (1 - \alpha) \|z_0\|^2. \quad (3.7)$$

On the other hand, by Lemma 2, we have

$$\|b\|^2 = \|h(z_0)\|^2 - \frac{\|z_0\|^2}{4\beta^2}(1 + \cos \theta)^2, \quad (3.8)$$

and  $\|h(z_0)\| \geq \frac{\|z_0\|}{2\beta} |1 + e^{i\theta}| = \frac{\|z_0\|}{2\beta} \sqrt{2 + 2\cos \theta}$ , and

$$\begin{aligned} m e^{-i\theta} \langle h(z_0), z_0 \rangle &= \langle e^{-i\theta} h(z_0), mz_0 \rangle \\ &= \langle e^{-i\theta} h(z_0), e^{i\theta} \overline{Dh(z_0)}(z_0) \rangle + \langle e^{-i\theta} h(z_0), e^{-i\theta} h(z_0) \rangle \\ &= e^{-i2\theta} \langle Dh(z_0)h(z_0), z_0 \rangle + \|h(z_0)\|^2, \end{aligned}$$

so that

$$\begin{aligned} \operatorname{Re} [e^{-i2\theta} \langle Dh(z_0)h(z_0), z_0 \rangle] &= m \operatorname{Re} [e^{-i\theta} \langle h(z_0), z_0 \rangle] - \|h(z_0)\|^2 \\ &= \frac{m\|z_0\|^2}{2\beta} (1 + \cos \theta) - \|h(z_0)\|^2. \end{aligned}$$

By computing the Frechet derivatives for both sides of equation  $Df(z)h(z) = f(z)$ , we obtain

$$[Df(z)]^{-1} D^2 f(z)(h(z), h(z)) + Dh(z)h(z) = h(z),$$

and

$$[Df(z)]^{-1} D^2 f(z)(h(z), z) + Dh(z)(z) = z,$$

thus we can obtain from the above equalities that

$$\begin{aligned} \operatorname{Re}([Df(z_0)]^{-1} D^2 f(z_0)(b, b), z_0) &= \operatorname{Re}[e^{-i2\theta} \langle [Df(z_0)]^{-1} D^2 f(z_0)(h(z_0), h(z_0)), z_0 \rangle \\ &\quad - \frac{1 + \cos \theta}{\beta} \operatorname{Re}[e^{-i\theta} \langle [Df(z_0)]^{-1} D^2 f(z_0)(h(z_0), z_0), z_0 \rangle] \\ &\quad + \frac{(1 + \cos \theta)^2}{4\beta^2} \operatorname{Re}[e^{-i\theta} \langle [Df(z_0)]^{-1} D^2 f(z_0)(z_0, z_0), z_0 \rangle] \\ &= -\operatorname{Re}[e^{-i2\theta} \langle Dh(z_0)h(z_0), z_0 \rangle] + \operatorname{Re}[e^{-i2\theta} \langle h(z_0), z_0 \rangle] \\ &\quad + \frac{1 + \cos \theta}{\beta} \{ \operatorname{Re}[e^{-i\theta} \langle Dh(z_0)(z_0), z_0 \rangle] - \operatorname{Re}[e^{-i\theta} \langle z_0, z_0 \rangle] \} \\ &\quad - \frac{(1 + \cos \theta)^2}{4\beta^2} \operatorname{Re}[e^{-i\theta} \langle [Df(z_0)]^{-1} D^2 f(z_0)(iz_0, iz_0), z_0 \rangle] \\ &\geq -\frac{m\|z_0\|^2}{2\beta} (1 + \cos \theta) + \|h(z_0)\|^2 - \frac{\|z_0\|^2}{2\beta} (1 + \cos \theta) \\ &\quad + \frac{\|z_0\|^2}{2\beta^2} (1 + \cos \theta)(\cos \theta + 1 + t) + \frac{(1 + \cos \theta)^2}{4\beta^2} (\alpha - 1) \|z_0\|^2 \\ &\geq (1 - \alpha) \|b\|^2 + \alpha \|h(z_0)\|^2 - \alpha \frac{\|z_0\|^2}{4\beta^2} (2 + 2 \cos \theta) \\ &\quad + \frac{1 + \cos \theta}{4\beta^2} \|z_0\|^2 (2\alpha - 2\beta + \frac{1 - \beta}{\beta}) \\ &\geq (1 - \alpha) \|b\|^2, \end{aligned}$$

which contradicts (3.6). Hence  $f \in S^*(B, \beta)$ , and the proof is complete.

**Remark 4.** Setting  $X = \mathbb{C}^n$  in Theorem 4, we obtain the related result in [7, 8, 15]

By applying the growth theorem[5] of starlike mappings of order  $\rho$  and Theorem 4, we have the following corollary.

**Corollary 3.** Let  $0 \leq \alpha < 1, \beta = \beta(\alpha) = \frac{2\alpha - 1 + \sqrt{(2\alpha - 1)^2 + 8}}{4}$ . If  $f(z) \in K(B, \alpha)$ , then for  $z \in B$ , we have

$$\frac{\|z\|}{(1 + \|z\|)^{2(1-\beta)}} \leq \|f(z)\| \leq \frac{\|z\|}{(1 - \|z\|)^{2(1-\beta)}},$$

and  $f(B) \supset \frac{1}{2^{2(1-\beta)}}B$ .

**Remark 5.** Setting  $\alpha = 0$  in Corollary 3, we obtain the growth theorem of convex mappings[1, 3].

Finally, we introduce a linear operator[16] in purpose to construct some concrete examples of biholomorphic convex mappings of order  $\alpha$  on  $B$  in a Hilbert space  $X$ .

Let

$$H(U) = \{f : U \rightarrow \mathbb{C} \text{ are analytic in } U \text{ with } f(0) = 0, f'(0) = 1\},$$

then

$$f \in K(\alpha) \iff f \in H(U) \text{ and } \operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha \text{ for all } z \in U.$$

Let

$$SK(B, \alpha) = \{f \in N(B) : \|Df(z)^{-1}D^2f(z)(\cdot, \cdot)\| \leq 1 - \alpha \text{ for all } z \in B\}.$$

From Theorem 1, we have  $SK(B, \alpha) \subset K(B, \alpha)$ ,  $SK(U, \alpha) \subset K(\alpha)$  and

$$SK(U, \alpha) = \left\{ f \in H(U) : \left| \frac{f''(z)}{f'(z)} \right| \leq 1 - \alpha \text{ for all } z \in U \right\}.$$

Let  $m$  be a positive integer and  $\dim X \geq m \geq 2$ . Then there exist  $u_1, u_2, \dots, u_m \in X$  with  $\|u_j\| = 1 (j = 1, 2, \dots, m)$  such that  $\langle u_j, u_k \rangle = 0 (j \neq k)$ . For  $g_1, g_2, \dots, g_m \in H(U)$ , we define the operator  $\Phi$  as

$$\Phi_{u_1, u_2, \dots, u_m}(g_1, g_2, \dots, g_m)(z) = z - \sum_{j=1}^m \langle z, u_j \rangle u_j + \sum_{j=1}^m g_j(\langle z, u_j \rangle) u_j \quad (3.9)$$

for  $z \in B$ .

**Theorem 5.** Suppose that  $0 \leq \alpha < 1$ ,  $\Phi_{u_1, u_2, \dots, u_m}(g_1, g_2, \dots, g_m)$  is defined by (3.9), where  $g_1, g_2, \dots, g_m \in H(U)$  are locally univalent functions on  $\Delta$ .

(1) If  $\Phi_{u_1, \dots, u_m}(g_1, \dots, g_m) \in K(B, \alpha)$ , then  $g_1, g_2, \dots, g_m \in K(\alpha)$ .

(2) If  $h(\xi) = \begin{cases} \frac{1-(1-\xi)^{2\alpha-1}}{2\alpha-1}, & \alpha \neq \frac{1}{2} \\ -\ln(1-\xi), & \alpha = \frac{1}{2} \end{cases}$ , then  $h \in K(\alpha)$ , but  $\Phi_{u_1, \dots, u_m}(h, \dots, h) \notin K(B, \alpha)$ .

(3)  $\Phi_{u_1, u_2, \dots, u_m}(g_1, g_2, \dots, g_m) \in SK(B, \alpha)$  if and only if  $g_1, g_2, \dots, g_m \in SK(U, \alpha)$ .

**Proof.** Let  $f(z) = \Phi_{u_1, u_2, \dots, u_m}(g_1, g_2, \dots, g_m)(z)$ , where  $g_1, g_2, \dots, g_m \in H(U)$  are locally univalent functions on  $U$ . By some straightforward computations, we obtain

$$\begin{aligned} Df(z) &= I - \sum_{j=1}^m \langle \cdot, u_j \rangle u_j + \sum_{j=1}^m g'_j(\langle z, u_j \rangle) \langle \cdot, u_j \rangle u_j, \\ Df(z)^{-1} &= I - \sum_{j=1}^m \left( 1 - \frac{1}{g'_j(\langle z, u_j \rangle)} \right) \langle \cdot, u_j \rangle u_j, \\ D^2f(z)(x, x) &= \sum_{j=1}^m g''_j(\langle z, u_j \rangle) [\langle x, u_j \rangle]^2 u_j \end{aligned}$$

for  $z \in B$  and  $x \in X$ . Hence we have

$$Df(z)^{-1}D^2f(z)(x, x) = \sum_{j=1}^m \frac{g_j''(\langle z, u_j \rangle)}{g_j'(\langle z, u_j \rangle)} [\langle x, u_j \rangle]^2 u_j. \quad (3.10)$$

(1) Assume that  $f = \Phi_{u_1, u_2, \dots, u_m}(g_1, g_2, \dots, g_m) \in K(B)$ , for every  $\xi \in U \setminus \{0\}$  and  $k$  fixed, we let  $z = \xi u_k$  and  $x = i\xi u_k$ , then  $\operatorname{Re}\langle x, z \rangle = \operatorname{Re}\{i|\xi|^2\} = 0$  and  $z \in B \setminus \{0\}$ . Note that  $\langle u_j, u_k \rangle = 0 (j \neq k)$ , from (3.10), we obtain

$$\begin{aligned} & \|x\|^2 - \operatorname{Re}\langle Df(z)^{-1}D^2f(z)(x, x), z \rangle \\ &= |\xi|^2 + |\xi|^2 \operatorname{Re}\left\{ \frac{\xi g_k''(\xi)}{g_k'(\xi)} \right\} = |\xi|^2 \operatorname{Re}\left\{ 1 + \frac{\xi g_k''(\xi)}{g_k'(\xi)} \right\} > \alpha \|x\|^2 = \alpha |\xi|^2 \end{aligned}$$

for  $\xi \in U \setminus \{0\}$ . This implies  $g_k \in K(\alpha)$  for  $k = 1, 2, \dots, m$ .

(2) A simple computation yields

$$\operatorname{Re}\left\{ 1 + \frac{\xi h''(\xi)}{h'(\xi)} \right\} = \alpha + (1 - \alpha) \operatorname{Re}\left\{ \frac{1 + \xi}{1 - \xi} \right\} > \alpha$$

for all  $\xi \in U$ . It follows  $h \in K(\alpha)$ .

$$\text{Let } \sqrt{\frac{1 - \alpha}{2}} < a < 1, \frac{\sqrt{1 - \alpha}}{\sqrt{2}a} < r < 1,$$

$$z = ru_1 + a\sqrt{1 - r^2}u_2 \quad \text{and} \quad x = a\sqrt{1 - r^2}u_1 - ru_2.$$

Then we have  $\|x\|^2 = a^2(1 - r^2) + r^2 > 0$ ,  $\operatorname{Re}\langle x, z \rangle = 0$  and

$$0 < \|z\|^2 = r^2 + a^2(1 - r^2) < 1.$$

Notice that  $\langle x, u_j \rangle = 0 (j \geq 3)$ , from (3.10), we obtain

$$\begin{aligned} & \|x\|^2 - \operatorname{Re}\langle DF(z)^{-1}D^2F(z)(x, x), z \rangle \\ &= \|x\|^2 - \operatorname{Re}\left\{ \frac{2}{1 - \langle z, u_1 \rangle} [\langle x, u_1 \rangle]^2 \langle u_1, z \rangle + \frac{2}{1 - \langle z, u_2 \rangle} [\langle x, u_2 \rangle]^2 \langle u_2, z \rangle \right\} \\ &= a^2(1 - r^2) + r^2 - \left\{ \frac{2r}{1 - r} a^2(1 - r^2) + \frac{2a\sqrt{1 - r^2}}{1 - a\sqrt{1 - r^2}} r^2 \right\} \\ &< 1 - 2r^2a^2 < \alpha, \end{aligned}$$

where  $F = \Phi_{u_1, u_2, \dots, m}(h, h, \dots, h)$ . By Corollary 1, we have  $F \notin K(B, \alpha)$ .

(3) Assume that  $g_1, g_2, \dots, g_m \in SK(U, \alpha)$ ,  $f = \Phi_{u_1, u_2, \dots, u_m}(g_1, g_2, \dots, g_m)$ , from (3.10), we obtain

$$\begin{aligned} \left\| Df(z)^{-1}D^2f(z)(x, x) \right\| &= \left\| \sum_{j=1}^m \frac{g_j''(\langle z, u_j \rangle)}{g_j'(\langle z, u_j \rangle)} [\langle x, u_j \rangle]^2 u_j \right\| \\ &\leq \sum_{j=1}^m \left| \frac{g_j''(\langle z, u_j \rangle)}{g_j'(\langle z, u_j \rangle)} \right| |\langle x, u_j \rangle|^2 \\ &\leq (1 - \alpha) \sum_{j=1}^m |\langle x, u_j \rangle|^2 \end{aligned} \quad (3.11)$$

for  $z \in B$  and  $x \in X$ .

Fix  $x \in X$ , let  $x_0 = \sum_{j=1}^m \langle x, u_j \rangle u_j$ , a simple computation yields

$$\langle x - x_0, u_j \rangle = \langle x, u_j \rangle - \sum_{k=1}^m \langle x, u_k \rangle \langle u_k, u_j \rangle = \langle x, u_j \rangle - \langle x, u_j \rangle = 0,$$

for  $j = 1, 2, \dots, m$ . This leads to  $\langle x - x_0, x_0 \rangle = 0$ . Hence we conclude that

$$\begin{aligned} \|x\|^2 &= \|(x - x_0) + x_0\|^2 = \|x - x_0\|^2 + \|x_0\|^2 \\ &= \|x - x_0\|^2 + \sum_{j=1}^m |\langle x, u_j \rangle|^2 \\ &\geq \sum_{j=1}^m |\langle x, u_j \rangle|^2. \end{aligned} \tag{3.12}$$

From (3.11) and (3.12), we obtain

$$\left\| Df(z)^{-1} D^2 f(z)(x, x) \right\| \leq (1 - \alpha) \sum_{j=1}^m |\langle x, u_j \rangle|^2 \leq (1 - \alpha) \|x\|^2 \leq 1 - \alpha$$

for all  $z \in B$  and  $x \in X$  with  $\|x\| = 1$ . Since  $X$  is a Hilbert space, by the result in [13](see P.342), we have

$$\begin{aligned} \|Df(z)^{-1} D^2 f(z)(\cdot, \cdot)\| &= \sup_{\|x\|=1, \|y\|=1} \|Df(z)^{-1} D^2 f(z)(x, y)\| \\ &= \sup_{\|x\|=1} \|Df(z)^{-1} D^2 f(z)(x, x)\| \leq 1 - \alpha. \end{aligned}$$

It follows that  $f \in SK(B, \alpha)$ .

Conversely, suppose that  $f = \Phi_{u_1, u_2, \dots, u_m}(g_1, g_2, \dots, g_m) \in SK(B, \alpha)$ . For every  $\xi \in U$  and  $k$  fixed ( $1 \leq k \leq m$ ), we let  $z = \xi u_k$  and  $x = u_k$ , then we have  $z \in B$ ,  $\langle z, u_k \rangle = \xi$  and  $\|x\| = 1$ . Note that  $\langle u_j, u_k \rangle = 0$  ( $j \neq k$ ) and  $\|u_k\| = 1$ , from (3.10), we obtain

$$\left| \frac{g_k''(\xi)}{g_k'(\xi)} \right| = \|Df(z)^{-1} D^2 f(z)(x, x)\| \leq \|Df(z)^{-1} D^2 f(z)(\cdot, \cdot)\| \|x\|^2 \leq 1 - \alpha$$

for  $\xi \in U$ . That is,  $g_k \in SK(U, \alpha)$  for  $k = 1, 2, \dots, m$ , and the proof is complete.

**Remark 6.** Let  $X = \mathbb{C}^n$ ,  $\alpha = 0$ . If we choose  $u_1 = (1, 0, \dots, 0)$ ,  $u_2 = (0, 1, \dots, 0)$ ,  $\dots$ ,  $u_n = (0, 0, \dots, 1) \in \mathbb{C}^n$ , then we have  $z = \sum_{j=1}^n \langle z, u_j \rangle u_j$  for  $z \in \mathbb{C}^n$ . From Theorem 5, we obtain a result, which is Theorem 3 and Theorem 4 in [14] for case  $p = 2$ . Part (2) was obtained by Roper and Suffridge [11], [12] using a different method.

**Example 3.** Let  $0 \leq \alpha < 1$ ,  $\dim X \geq m \geq 2$  and  $\lambda_j \in \mathbb{C}$  with  $\lambda_j \neq 0$  ( $j = 1, 2, \dots, m$ ), then

$$f(z) = z - \sum_{j=1}^m \langle z, u_j \rangle u_j + \sum_{j=1}^m \frac{e^{\lambda_j \langle z, u_j \rangle} - 1}{\lambda_j} u_j \in K(B, \alpha) \iff |\lambda_j| \leq 1 - \alpha \quad (j = 1, 2, \dots, m),$$

where  $u_j \in X$  with  $\|u_j\| = 1$  such that  $\langle u_j, u_k \rangle = 0 (j, k = 1, 2, \dots, m, j \neq k)$ .

**Proof.** If  $|\lambda_j| \leq 1 (j = 1, 2, \dots, m)$ , setting  $g_j(\xi) = \frac{e^{\lambda_j \xi} - 1}{\lambda_j}$ , then we have that  $g_j$  is analytic on  $U$  with  $g_j(0) = 0, g'_j(0) = 1$  such that  $\frac{g''_j(\xi)}{g'_j(\xi)} = \lambda_j$  for  $\xi \in U (j = 1, 2, \dots, m)$ . Hence  $g_j \in SK(U)$ . From Theorem 5, we obtain  $f \in SK(B, \alpha)$ .

Conversely, we shall prove that  $|\lambda_j| \leq 1 - \alpha$  for all  $j = 1, 2, \dots, m$  when  $f$  is a biholomorphic convex mapping of order  $\alpha$  on  $B$ .

If not, then there exists  $k$  such that  $|\lambda_k| > 1 - \alpha$ . Let  $\frac{1 - \alpha}{|\lambda_k|} < r < 1, \theta = \arg \lambda_k, z_0 = -re^{-i\theta}u_k$  and  $x = ie^{-i\theta}u_k$ , then  $\|x\| = 1, \operatorname{Re}\langle x, z_0 \rangle = \operatorname{Re}\{-ir\} = 0$ . Using the fact that  $\langle u_j, u_k \rangle = 0 (j \neq k)$ , from (3.10), we obtain

$$\begin{aligned} Df(z_0)^{-1}D^2f(z_0)(x, x) &= \sum_{j=1}^m \frac{g''_j(\langle z_0, u_j \rangle)}{g'_j(\langle z_0, u_j \rangle)} [\langle x, u_j \rangle]^2 u_j \\ &= \lambda_k [\langle x, u_k \rangle]^2 u_k = -|\lambda_k| e^{-i\theta} u_k. \end{aligned}$$

Hence we have

$$\|x\|^2 - \operatorname{Re}\langle Df(z_0)^{-1}D^2f(z_0)(x, x), z_0 \rangle = 1 - r|\lambda_k| < \alpha,$$

which contradicts (2.2). This completes the proof.

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